Real Analysis tutorial class 3

Example 0.1. *There exists a open set $G \subset \mathbb{R}$ with $m(\partial G) > 0$.*

Proof. Recall the construction of cantor set. Instead of removing 1/3 of the middle of each interval, we remove a $\alpha/3$ part of the middle interval where $\alpha \in (0,1)$. At the n-th stage, we remove 2^{n-1} pieces of interval. Each of those is of length $\alpha 3^{-n}$. The resulting "cantor set" is still uncountable and nowhere dense, but of measure $1-\alpha$. By taking its complement, we obtain the desired open set. □

Lemma 0.2. *Let $E_i \in M$ with $\sum_{i=1}^{\infty} m(E_i) < \infty$. Then at most all x belongs to at most finitely many E_k.*

Proof. Denote

$$E = \{x : x \text{ belongs to at most finitely many } E_k\}$$

We show that E^c is of measure zero. $x \notin E$ if and only if for any $n \in \mathbb{N}$, $\exists k \geq n$ such that $x \in E_k$. Hence,

$$E^c = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k = \bigcap_{n \in \mathbb{N}} A_n.$$

A_n is descending and $m(A_1) < \infty$. By continuity of measure,

$$m(E^c) = \lim_{n\to\infty} m(A_n) \leq \lim_{n\to\infty} \sum_{k=n}^{\infty} m(E_k) = 0.$$

□